

Tight Lower Bound on the Geometric Discord: A Measure of the Quantumness

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A general state of an $m \otimes n$ system is a classical-quantum state if and only if its associated left-correlation matrix has rank no larger than $m - 1$. Based on this condition, a computable measure of quantum discord is presented, which coincides with the tight lower bound on the geometric measure of discord. Therefore such obtained tight lower bound fully captures the quantum correlation of a bipartite system, so it can be used as a measure of discord in its own right. Accordingly, a vanishing tight lower bound on the geometric discord is a necessary and sufficient condition for a state to be zero-discord. We present an alternative form for the geometric discord and provide a comparison between the geometric discord and our computable measure of quantum correlation.

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I. INTRODUCTION

Quantum discord represents a new type of quantum correlation which looks at the correlations from a new perspective, i.e. measurement theory, different than the entanglement-separability paradigm [1, 2]. Hence, there exist separable (not-entangled) states which have non-zero discord such that one can employ these separable states as a resource to enhance the quality of quantum information and computation processing [3, 4]. Nowadays, quantum discord became a subject of intensive study in different contexts [5] and different versions of quantum discord and their measures have been introduced and analyzed [5, 6]. Since the evaluation of quantum discord involves an optimization procedure, almost all quantum discord measures are very difficult to calculate analytically and quantum discord was analytically computed only for a few families of two-qubit states [7] and for some reduced two-qubit states of pure three-qubit states and also a class of rank-2 mixed state of $4 \otimes 2$ systems [8]. Among the various measures of quantum discord, the geometric discord, has been firstly proposed by Dakic *et al.*, is a simple and intuitive quantifier of general non-classical correlations [9]. Geometric discord is defined as the squared Hilbert-Schmidt distance between the state of the quantum system and the closest zero-discord state. For a bipartite state ρ on $\mathcal{H}^A \otimes \mathcal{H}^B$, with $\dim \mathcal{H}^A = m$ and $\dim \mathcal{H}^B = n$, the geometric discord is defined by [9]

$$D_G(\rho) = \min_{\chi \in \Omega_0} \|\rho - \chi\|^2, \quad (1)$$

where Ω_0 denotes the set of all zero-discord states and $\|X - Y\|^2 = \text{Tr}(X - Y)^2$ is the 2-norm or square norm in the Hilbert-Schmidt space. This quantity vanishes on the classical-quantum states. Dakic *et al.* also obtained a closed formula for the geometric discord of an arbitrary two-qubit state in terms of coherence vectors and correlation matrix of the state. Furthermore, an exact expression for the pure $m \otimes m$ states and arbitrary $2 \otimes n$ states are obtained [10, 11].

An alternative form for the geometric discord is intro-

duced by Luo and Fu [10]

$$D_G(\rho) = \min_{\Pi^A} \|\rho - \Pi^A(\rho)\|^2, \quad (2)$$

where the minimum is taken over all von Neumann measurements $\Pi^A = \{\Pi_k^A\}_{k=1}^m$ on \mathcal{H}^A , and $\Pi^A(\rho) = \sum_{k=1}^m (\Pi_k^A \otimes \mathbb{I}) \rho (\Pi_k^A \otimes \mathbb{I})$ with \mathbb{I} as the identity operator on the appropriate space. They have also shown that Eq. (2) is equivalent to [10]

$$D_G(\rho) = \text{Tr}(CC^t) - \max_A \text{Tr}(ACC^t A^t), \quad (3)$$

where t denotes transpose, and $C = (c_{ij})$ is an $m^2 \times n^2$ -dimensional matrix defined by

$$\rho = \sum_{i=0}^{m^2-1} \sum_{j=0}^{n^2-1} c_{ij} X_i \otimes Y_j, \quad (4)$$

with $\{X_i\}_{i=1}^{m^2-1}$ and $\{Y_j\}_{j=1}^{n^2-1}$ as the sets of Hermitian operators which constitute orthonormal basis for $SU(m)$ and $SU(n)$ algebra, respectively, i.e.

$$\text{Tr}(X_i X_{i'}) = \delta_{ii'}, \quad \text{Tr}(Y_j Y_{j'}) = \delta_{jj'}. \quad (5)$$

In Eq. (3), the maximum is taken over all $m \times m^2$ -dimensional matrices $A = (a_{ki})$ such that

$$a_{ki} = \text{Tr}(|k\rangle\langle k| X_i) = \langle k| X_i |k\rangle, \quad (6)$$

where $\{|k\rangle\}_{k=1}^m$ is any orthonormal base for \mathcal{H}^A . Based on the definition (3), Rana *et al.* [12] and Hassan *et al.* [13] have obtained a tight lower bound on the geometric discord. Let $\{\hat{\lambda}_i^A\}_{i=1}^{m^2-1}$ and $\{\hat{\lambda}_j^B\}_{j=1}^{n^2-1}$ be generators of $SU(m)$ and $SU(n)$, respectively, fulfilling the following relations

$$\text{Tr} \hat{\lambda}_i^s = 0, \quad \text{Tr}(\hat{\lambda}_i^s \hat{\lambda}_j^s) = 2\delta_{ij}, \quad s = A, B. \quad (7)$$

Then a general bipartite state ρ on $\mathcal{H}^A \otimes \mathcal{H}^B$ can be written in this basis as

$$\rho = \frac{1}{mn} \left(\mathbb{I} \otimes \mathbb{I} + \vec{x} \cdot \hat{\lambda}^A \otimes \mathbb{I} + \mathbb{I} \otimes \vec{y} \cdot \hat{\lambda}^B + \sum_{i=1}^{m^2-1} \sum_{j=1}^{n^2-1} t_{ij} \hat{\lambda}_i^A \otimes \hat{\lambda}_j^B \right). \quad (8)$$

Here \mathbb{I} stands for the identity operator, $\vec{x} = (x_1, \dots, x_{m^2-1})^t$ and $\vec{y} = (y_1, \dots, y_{n^2-1})^t$ are local coherence vectors of the subsystems A and B , respectively

$$x_i = \frac{m}{2} \text{Tr}[(\hat{\lambda}_i^A \otimes \mathbb{I})\rho], \quad y_j = \frac{n}{2} \text{Tr}[(\mathbb{I} \otimes \hat{\lambda}_j^B)\rho], \quad (9)$$

and $T = (t_{ij})$ is the correlation matrix

$$t_{ij} = \frac{mn}{4} \text{Tr}[(\hat{\lambda}_i^A \otimes \hat{\lambda}_j^B)\rho]. \quad (10)$$

Comparing the two forms of ρ given by Eqs. (4) and (8), we find that

$$C = \frac{1}{\sqrt{mn}} \begin{pmatrix} 1 & \sqrt{\frac{2}{n}} \vec{y}^t \\ \sqrt{\frac{2}{m}} \vec{x} & \frac{2}{\sqrt{mn}} T \end{pmatrix}. \quad (11)$$

The authors of [12, 13] have shown that the geometric discord of ρ is lower bounded as

$$\begin{aligned} D_G(\rho) &\geq \frac{2}{m^2 n} \left(\|\vec{x}\|^2 + \frac{2}{n} \|T\|^2 - \sum_{k=1}^{m-1} \eta_k^\downarrow \right) \\ &= \frac{2}{m^2 n} \sum_{k=m}^{m^2-1} \eta_k^\downarrow, \end{aligned} \quad (12)$$

where $\{\eta_k^\downarrow\}_{k=1}^{m^2-1}$ are eigenvalues of

$$G := \vec{x}\vec{x}^t + \frac{2}{n} T T^t, \quad (13)$$

in nonincreasing order. Remarkably, the above lower bound on the geometric discord is tight in the sense that for $m \otimes m$ Werner and isotropic states, the above lower bound are achieved [10, 12]. Furthermore, for an arbitrary state of $2 \otimes n$ systems, the geometric discord coincides with this lower bound [11, 14]. In appendix A, we provide an alternative form for the geometric measure which can be used to obtain the lower bound (12) as well as an exact solution for states of $2 \otimes n$ systems.

Based on the rank of the correlation matrix, Dakic *et al.* [9] obtained a simple necessary condition for a general bipartite state to be zero-discord. A necessary and sufficient condition for a two-qubit state to be zero-discord is obtained by Lu *et al.* [15]. Their condition is related to the existence of a unit vector $\hat{n} \in \mathbb{R}^3$ satisfying the following conditions

$$\hat{n} \hat{n}^t \vec{x} = \vec{x}, \quad \hat{n} \hat{n}^t T = T, \quad (14)$$

where \vec{x} denotes coherence vector of the subsystem A , and T is the correlation matrix of ρ in Bloch representation. Accordingly, a two-qubit state is of zero-discord if and only if either $T = 0$, or $\text{rank}(T) = 1$ and \vec{x} belongs to the range of T . Recently, Zhou *et al.* [16] introduced a criterion tensor as

$$\Lambda = \left(\frac{4}{mn} \right)^2 (T T^t - y^2 \vec{x} \vec{x}^t), \quad (15)$$

and showed that a necessary and sufficient condition for a bipartite state to be zero-discord is $\text{rank}(\Lambda) \leq m - 1$. Remarkably, the above criterion is based on the extended version of Eq. (14) as [16]

$$P \vec{x} = \vec{x}, \quad P T = T, \quad (16)$$

where P is an $(m-1)$ -dimensional projection operator on space \mathbb{R}^{m^2-1} . Based on the criterion tensor, the authors of [16] proposed a measure of the quantum correlation as

$$Q(\rho) = \frac{1}{4} \sum_{k=m}^{m^2-1} |\Lambda_k^\downarrow|, \quad (17)$$

where $\{\Lambda_k^\downarrow\}_{k=1}^{m^2-1}$ are eigenvalues of the criterion tensor (15) in nonincreasing order. They have also shown that in some particular cases their measure coincides with the geometric measure of quantum discord.

In this paper we use the conditions given by Eq. (16) and propose a geometric way of quantifying quantum discord. The optimization involved in the definition can be solved analytically, leading therefore to a closed form for the discord. Remarkably, such defined measure of discord equals to the tight lower bound on the geometric discord given in (12). This suggest that such obtained lower bound fully captures the quantum correlation and can be used as a measure of discord in its own right.

The paper is organized as follows. In section II, we review some properties of coherence vectors of an arbitrary set of von Neumann projection operators on \mathcal{H}^A . The necessary and sufficient condition for a state to be zero-discord is also given in section II. Section III is devoted to the definition of the new measure of quantumness. In this section we also present some properties of the new measure and provide a comparison of this measure with the geometric measure and the measure given in Eq. (17). The paper is concluded in section IV.

II. CHARACTERIZING CLASSICAL-QUANTUM STATES

A general density operator on \mathcal{H}^A can be written as

$$\rho^A = \frac{1}{m} \left(\mathbb{I} + \vec{x} \cdot \hat{\lambda}^A \right), \quad (18)$$

where (m^2-1) -dimensional vector $\vec{x} = (x_1, \dots, x_{m^2-1})^t$, with $x_i = \frac{m}{2} \text{Tr}(\hat{\lambda}_i^A \rho^A)$, is the so-called coherence vector of ρ^A . For further use, we give bellow some properties of coherence vectors of a set of orthonormal pure states. Let $\{|k\rangle\}_{k=1}^m$ be an arbitrary orthonormal base for \mathcal{H}^A and $\{\Pi_k^A = |k\rangle\langle k|\}_{k=1}^m$ denotes projectors on this base; then

$$\Pi_k^A \Pi_{k'}^A = \Pi_k^A \delta_{kk'}, \quad \sum_{k=1}^m \Pi_k^A = \mathbb{I}. \quad (19)$$

Now let $\vec{\alpha}_k \in \mathbb{R}^{m^2-1}$ denotes coherence vector corresponding to Π_k^A , i.e.

$$\Pi_k^A = \frac{1}{m} \left(\mathbb{I} + \vec{\alpha}_k \cdot \hat{\lambda}^A \right), \quad (20)$$

then the orthonormality and completeness conditions given in Eq. (19) require that $\{\vec{\alpha}_k\}_{k=1}^m$ fulfill the following two conditions

$$\vec{\alpha}_k \cdot \vec{\alpha}_{k'} = -\frac{m}{2} + \frac{m^2}{2} \delta_{kk'}, \quad \sum_{k=1}^m \vec{\alpha}_k = \vec{0}. \quad (21)$$

From the first relation above we find

$$|\vec{\alpha}_k| = \sqrt{\frac{m(m-1)}{2}}, \quad \cos \theta_{kk'} = \frac{-1}{m-1}, \quad (22)$$

where $\theta_{kk'}$ ($k \neq k'$) is the angle between a pair of coherence vectors $\vec{\alpha}_k$ and $\vec{\alpha}_{k'}$. This implies that the set of $(m^2 - 1)$ -dimensional coherence vectors $\{\vec{\alpha}_k\}_{k=1}^m$ corresponding to an orthonormal base forms an $(m - 1)$ -dimensional simplex. In what follows, we denote this kind of simplex by $\Delta_{\{\vec{\alpha}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$. Corresponding to any such defined simplex, the following lemma gives an $(m - 1)$ -dimensional projection operator on \mathbb{R}^{m^2-1} [16].

Lemma 1 *Any $(m - 1)$ -dimensional projection operator on $(m^2 - 1)$ -dimensional space \mathbb{R}^{m^2-1} can be represented by*

$$P = \frac{2}{m^2} \sum_{k=1}^m \vec{\alpha}_k (\vec{\alpha}_k)^t, \quad (23)$$

where $\{\vec{\alpha}_k\}_{k=1}^m$ are coherence vectors corresponding to orthonormal projections, satisfying Eqs. (21) and (22).

Proof First note that one can easily show that $P^\dagger = P$ and $P^2 = P$, so P is a projection operator. Since coherence vectors corresponding to orthonormal base make simplex $\Delta_{\{\vec{\alpha}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$, so P is an $(m - 1)$ -dimensional projection operator on \mathbb{R}^{m^2-1} or equivalently it is the unit operator on space \mathbb{R}^{m-1} .

Let us turn our attention on the bipartite state ρ on $\mathcal{H}^A \otimes \mathcal{H}^B$ and consider the set of zero-discord states. By definition, a bipartite state ρ is of zero-discord, i.e. classical-quantum state, if and only if there exists orthonormal base $\{|k\rangle\}_{k=1}^m$ of \mathcal{H}^A such that [1]

$$\rho = \sum_{k=1}^m p_k \Pi_k^A \otimes \rho_k^B, \quad (24)$$

where $\Pi_k^A = |k\rangle\langle k|$ and ρ_k^B is a state on \mathcal{H}^B . The following theorem gives a criterion for a state to be zero-discord [16].

Theorem 2 *A bipartite state ρ on the $\mathcal{H}^A \otimes \mathcal{H}^B$ is a zero-discord state, a classical-quantum state, if and only*

if there exists an $(m - 1)$ -dimensional projection operator P on the $(m^2 - 1)$ -dimensional space \mathbb{R}^{m^2-1} such that

$$P\vec{x} = \vec{x}, \quad PT = T, \quad (25)$$

where \vec{x} denotes coherence vector of party A, and T is the correlation matrix of ρ .

A proof of this theorem is given in appendix B (see also [16]). Let us mention here that conditions (25) can be written also as

$$PT = \mathcal{T}, \quad (26)$$

where \mathcal{T} is an $(m^2 - 1) \times m^2$ matrix, obtained by removing the first row of the $m^2 \times m^2$ matrix C of Eq. (11), i.e.

$$\mathcal{T} := \sqrt{\frac{2}{m^2 n}} \begin{pmatrix} \vec{x} & \sqrt{\frac{2}{n}} T \end{pmatrix}. \quad (27)$$

Since \mathcal{T} includes coherence vector \vec{x} of the subsystem A as well as the correlation matrix T of the bipartite system $A - B$, we call \mathcal{T} as the left-correlation matrix associated to the state ρ .

As an example, let us consider the case of two-qubit system. In this case a general zero-discord state χ is characterized by $\vec{x} = (p_1 - p_2)\hat{n}$, $\vec{y} = (p_1\vec{\xi}_1 + p_2\vec{\xi}_2)$, and $T = \hat{n}(p_1\vec{\xi}_1 - p_2\vec{\xi}_2)^t$, where p_1, p_2 are probabilities with $p_1 + p_2 = 1$, \hat{n} is a unit vector, and $\vec{\xi}_1, \vec{\xi}_2$ are coherence vectors of the subsystem B. Evidently, the zero-discord condition (14) is satisfied. In the following we show that the above theorem provides a necessary and sufficient condition for a bipartite state ρ to be zero-discord [9, 16].

Corollary 3 *A bipartite state ρ with the left-correlation matrix \mathcal{T} , associated to the local coherence vector \vec{x} and correlation matrix T , is a classical-quantum state, i.e. zero-discord state, if and only if*

$$\text{rank}(\mathcal{T}\mathcal{T}^t) \leq m - 1.$$

Equivalently, one can say that ρ is a zero-discord state if and only if one of the following conditions is satisfied

- (i) $\text{rank}(TT^t) \leq m - 2$,
- (ii) $\text{rank}(TT^t) \leq m - 1$, and $\vec{x} \in \text{R}(TT^t)$,

where $\text{R}(M)$ denotes range of the matrix M .

III. QUANTIFYING QUANTUM DISCORD

Theorem 2 allows us to introduce a new measure of quantum discord. Since conditions (25) give necessary and sufficient conditions for a state to be zero-discord, therefore measuring any violation of these conditions can be used to quantify discord. Accordingly, we use the degree to which the above conditions fail to be satisfied as a measure of discord. Here we propose the following measure of quantum discord

Proposition 4 For a given bipartite state ρ with the left-correlation matrix \mathcal{T} , associated to the local coherence vector \vec{x} and correlation matrix T , we propose the measure of quantum discord as

$$D_P(\rho) = \min_P \|\mathcal{T} - P\mathcal{T}\|^2 \\ = \frac{2}{m^2 n} \min_P \{\|\vec{x} - P\vec{x}\|^2 + \frac{2}{n} \|T - PT\|^2\}, \quad (28)$$

where $\|\cdot\|^2$ may be any reasonable norm on the space of states and the minimum is taken over all $(m-1)$ -dimensional projection operators P on \mathbb{R}^{m^2-1} . Here we take $\|\cdot\|^2$ to be the square norm in the Hilbert-Schmidt space.

As we show below, minimization involved in the definition given above can be solved exactly for arbitrary $m \otimes n$ states, giving therefore a closed form of expression for discord. To see this, we write a general $(m-1)$ -dimensional projection operator P on \mathbb{R}^{m^2-1} as $P = \sum_{i=1}^{m-1} \hat{n}_i \hat{n}_i^t$, where $\{\hat{n}_i\}_{i=1}^{m-1}$ is an arbitrary orthonormal base for $(m-1)$ -dimensional subspace of \mathbb{R}^{m^2-1} , i.e. $\hat{n}_i^t \hat{n}_j = \delta_{ij}$. Therefore

$$D_P(\rho) = \frac{2}{m^2 n} \min_P \{\|P\vec{x} - \vec{x}\|^2 + \frac{2}{n} \|PT - T\|^2\} \\ = \frac{2}{m^2 n} \left[\text{Tr}G - \max_P \text{Tr}(\vec{x}^t P \vec{x} + \frac{2}{n} T^t P T) \right] \\ = \frac{2}{m^2 n} \left[\text{Tr}G - \max_{\{\hat{n}_i\}} \text{Tr} \sum_{i=1}^{m-1} \hat{n}_i^t G \hat{n}_i \right], \quad (29)$$

where G is defined by Eq. (13). Using the fact that

$$\mathcal{T}\mathcal{T}^t = \frac{2}{m^2 n} \left(\vec{x}\vec{x}^t + \frac{2}{n} T T^t \right) = \frac{2}{m^2 n} G, \quad (30)$$

we can rewrite Eq. (29) as

$$D_P(\rho) = \left[\text{Tr}(\mathcal{T}\mathcal{T}^t) - \max_{\{\hat{n}_i\}} \sum_{i=1}^{m-1} \hat{n}_i^t (\mathcal{T}\mathcal{T}^t) \hat{n}_i \right] \\ = \left[\text{Tr}(\mathcal{T}\mathcal{T}^t) - \sum_{k=1}^{m-1} \tau_k^\downarrow \right] = \sum_{k=m}^{m^2-1} \tau_k^\downarrow, \quad (31)$$

where $\{\tau_k^\downarrow\}_{k=1}^{m^2-1}$ are eigenvalues of $\mathcal{T}\mathcal{T}^t$ in nonincreasing order. Surprisingly, the above measure coincides with the tight lower bound on the geometric discord (12), therefore we have in general

$$D_P(\rho) \leq D_G(\rho). \quad (32)$$

This, particularly, implies that ρ is a zero-discord state if and only if the lower bound (12) vanishes. We can also define the total and classical correlations, respectively, as (see also [16])

$$\mathcal{I}_P(\rho) = \|\mathcal{T}\|^2 = \sum_{k=1}^{m^2-1} \tau_k^\downarrow, \quad (33)$$

$$\mathcal{C}_P(\rho) = \max_P \|P\mathcal{T}\|^2 = \sum_{k=1}^{m-1} \tau_k^\downarrow. \quad (34)$$

In view of this, the measure of quantum correlation reads as $D_P(\rho) = \mathcal{I}_P(\rho) - \mathcal{C}_P(\rho)$.

Let us mention some properties of the above defined measure of discord. (i) By definition, the above measure of discord vanishes only for zero-discord states. (ii) For any maximally entangled state $|\Psi\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^m |ii\rangle$, we have $D_P(\rho) = D_G(\rho) = \frac{m-1}{m}$, which achieves its maximum value. (iii) $D_P(\rho)$ is invariant under any local unitary operations U_1 and U_2 performed on \mathcal{H}^A and \mathcal{H}^B respectively, i.e. $D_P((U_1 \otimes U_2)\rho(U_1 \otimes U_2)^\dagger) = D_P(\rho)$ where $U_1 \in SU(m)$ and $U_2 \in SU(n)$. This follows from the fact the under such transformations, the coherence vectors \vec{x} , \vec{y} and the correlation matrix T transform as

$$\vec{x} \rightarrow O_1 \vec{x}, \quad \vec{y} \rightarrow O_2 \vec{y}, \quad T \rightarrow O_1 T O_2^t, \quad (35)$$

where O_1 corresponds to U_1 via $U_1(\vec{x} \cdot \hat{\lambda}^A)U_1^\dagger = (O_1 \vec{x}) \cdot \hat{\lambda}^A$ with $O_1 \in SO(m^2-1)$. A similar definition holds for O_2 . This leads to $G \rightarrow O_1 G O_1^t$, leaving eigenvalues of G invariant.

Note that the above measure is, in general, different from the measure of nonclassical correlation (17), proposed by Zhou *et al.* in Ref. [16]. More precisely, $Q(\rho)$ is obtained from the first $m(m-1)$ smaller eigenvalues of the criterion tensor Λ , but $D_P(\rho)$ is obtained from the first $m(m-1)$ smaller eigenvalues of the matrix $\mathcal{T}\mathcal{T}^t$, with \mathcal{T} as the left-correlation matrix. Evidently, when $\vec{x} = 0$ then $\mathcal{T}\mathcal{T}^t = \frac{1}{4}\Lambda$, so that the two definitions become identical. Moreover, in all cases that $m(m-1)$ smaller eigenvalues (counting multiplicity) of three matrices $\mathcal{T}\mathcal{T}^t$, $\frac{1}{4}\Lambda$, and $\frac{4}{m^2 n^2} T T^t$ coincide, two definitions give same results.

Note also the fact that the above measure of discord is in some sense similar to the geometric measure. Indeed, the geometric discord $D_G(\rho)$, as given in Eq. (2), is defined as the distance between a given ρ and the closest state $\Pi^A(\rho)$, for all von Neumann (projective) measurements $\Pi^A = \{\Pi_k^A\}_{k=1}^m$ acting on \mathcal{H}^A . On the other hand, $D_P(\rho)$ is defined as the distance between the left-correlation matrix \mathcal{T} associated to ρ and the closest left-correlation matrix $P(\mathcal{T})$, for all $(m-1)$ -dimensional projection operators P acting on \mathbb{R}^{m^2-1} . Clearly, difference arise from the level of calculation of the Hilbert-Schmidt distance; the former is in the level of density operator ρ but the later is in the level of left-correlation matrix \mathcal{T} . As it is shown in appendix A, an alternative form for the geometric discord is given by

$$D_G(\rho) = \frac{2}{m^2 n} \left[\text{Tr}G - \max_{\{\vec{\mu}_k\}} \sum_{k=1}^m \vec{\mu}_k^t G \vec{\mu}_k \right], \quad (36)$$

where maximum is taken over all simplexes $\Delta_{\{\vec{\mu}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$. A comparison of Eq. (31) with Eq. (36) shows that the calculation of $D_P(\rho)$ needs to perform optimization over $(m-1)$ -dimensional projection operators P , which can be solved exactly, but in calculation of $D_G(\rho)$ we have to make optimization over $(m-1)$ -dimensional simplexes $\Delta_{\{\vec{\mu}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$, where does not have an exact solution in

general. Two definitions become identical when $m = 2$, namely for $2 \otimes n$ systems. This happens because in case $m = 2$, calculation of the geometric discord leads to the problem of optimization over one-dimensional simplexes $\Delta_{\{\vec{\mu}_1, \vec{\mu}_2\} \in \mathbb{R}^3}^1$ with $\vec{\mu}_1 = -\vec{\mu}_2 = \frac{1}{\sqrt{2}}\vec{\alpha}_1$ and $|\vec{\alpha}_1| = 1$, which is the same as the problem of optimization over one-dimensional projection operators P . Surprisingly, as it is evident from Eqs. (28) and (36), both definitions are independent of the coherence vector \vec{y} of the second subsystem. We give below some illustrative examples.

$m \otimes m$ Werner states.— For the $m \otimes m$ Werner states

$$\rho = \frac{m-x}{m^3-m} \mathbb{I}_m + \frac{mx-1}{m^3-m} F, \quad x \in [-1, 1], \quad (37)$$

with $F = \sum_{k,l=1}^m |kl\rangle\langle lk|$, the geometric measure of discord is [10]

$$D_G(\rho) = \frac{(mx-1)^2}{m(m-1)(m+1)^2}. \quad (38)$$

On the other hand for these states $\vec{x} = \vec{y} = 0$ and

$$\mathcal{T}\mathcal{T}^t = \text{diag}\{\tau, \dots, \tau\}, \quad \text{with } \tau = \frac{(mx-1)^2}{m^2(m^2-1)^2}, \quad (39)$$

so that we get $D_P(\rho) = m(m-1)\tau = D_G(\rho)$.

$m \otimes m$ Isotropic States.— As the second example we consider $m \otimes m$ isotropic states defined by

$$\rho = \frac{1-x}{m^2-1} \mathbb{I}_m + \frac{m^2x-1}{m^2-1} |\psi\rangle\langle\psi|, \quad x \in [0, 1], \quad (40)$$

with $|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{k=1}^m |kk\rangle$. The geometric measure of discord is [10]

$$D_G(\rho) = \frac{(m^2x-1)^2}{m(m-1)(m+1)^2}. \quad (41)$$

On the other hand for these states $\vec{x} = \vec{y} = 0$ and

$$\mathcal{T}\mathcal{T}^t = \text{diag}\{\tau, \dots, \tau\}, \quad \text{with } \tau = \frac{(m^2x-1)^2}{m^2(m^2-1)^2}, \quad (42)$$

we obtain $D_P(\rho) = m(m-1)\tau = D_G(\rho)$.

Pure $m \otimes m$ states.— Finally, we consider an example of bipartite $m \otimes m$ pure state $|\Psi\rangle$, with the following Schmidt decomposition

$$|\Psi\rangle = \sum_{i=1}^m \sqrt{s_i} |i\rangle |i\rangle. \quad (43)$$

The geometric discord of this state is [11]

$$D_G(\Psi) = 1 - \sum_{i=1}^m s_i^2 = 1 - \text{Tr}(\rho^A)^2 = \frac{1}{2} C^2(\Psi), \quad (44)$$

where $\rho^A = \text{Tr}_B |\Psi\rangle\langle\Psi|$ is the reduced state of the subsystem A , and $C(\Psi)$ is the generalized concurrence of $|\Psi\rangle$ [18]. On the other hand, in order to evaluate $D_P(\Psi)$ we

have to find the local coherence vectors and the correlation matrix associated with $\rho = |\Psi\rangle\langle\Psi|$, we get

$$x_k = y_k = \frac{m}{2} \sum_{i=1}^m s_i \langle i | \hat{\lambda}_k | i \rangle, \quad (45)$$

$$\begin{aligned} t_{kl} &= \frac{m^2}{4} \sum_{i=1}^m \sum_{j=1}^m \sqrt{s_i s_j} \langle i | \hat{\lambda}_k | i \rangle \langle i | \hat{\lambda}_l^* | i \rangle \\ &= \frac{m^2}{4} \text{Tr} \left(\sqrt{\rho^A} \hat{\lambda}_k \sqrt{\rho^A} \hat{\lambda}_l^* \right), \end{aligned} \quad (46)$$

for $k, l = 1, \dots, m^2 - 1$, where $\{\hat{\lambda}_k\}_{k=1}^{m^2-1}$ are basis of $SU(m)$ algebra. If we choose the basis of $SU(m)$ in such a way that the first $m-1$ generators make the basis of its Cartan subalgebra, we get

$$x_k = y_k = \begin{cases} \frac{m(\sum_{i=1}^k s_i - k s_{k+1})}{\sqrt{2k(k+1)}}, & k = 1, \dots, m-1 \\ 0 & k = m, \dots, m^2-1 \end{cases} \quad (47)$$

$$T = \frac{m^2}{2} \begin{pmatrix} T_c & 0 \\ 0 & T_d \end{pmatrix}, \quad (48)$$

where T_c is an $(m-1) \times (m-1)$ symmetric matrix with

$$(T_c)_{kk} = \frac{1}{k(k+1)} \left(\sum_{i=1}^k s_i + k^2 s_{k+1} \right), \quad (49)$$

$$(T_c)_{k<l} = \frac{1}{\sqrt{k(k+1)l(l+1)}} \left(\sum_{i=1}^k s_i - k s_{k+1} \right), \quad (50)$$

and T_d is an $m(m-1) \times m(m-1)$ diagonal matrix such that $T_d = \text{diag}\{\pm\sqrt{s_1 s_2}, \pm\sqrt{s_1 s_3}, \dots, \pm\sqrt{s_{m^2-2} s_{m^2-1}}\}$. For maximally entangled states we have $s_i = \frac{1}{\sqrt{m}}$ for $i = 1, \dots, m$, leads to $\vec{x} = \vec{y} = \vec{0}$, $TT^t = \frac{m^2}{4} I_{m^2-1}$; so that $\mathcal{T}\mathcal{T}^t = \frac{1}{m^2} I_{m^2-1}$ and $D_P(\Psi) = D_G(\Psi) = \frac{m-1}{m}$.

IV. CONCLUSION

We have presented a measure of the quantumness of correlation for arbitrary bipartite states. Our measure is based on the necessary and sufficient condition for a state to be zero-discord. An analytical expression for this measure is given for any bipartite state. Interestingly, this measure equals to the tight lower bound on the geometric discord. We have shown that both geometric measure and our measure of quantum correlation are independent of the coherence vector of the second subsystem. We provide some examples and show that our measure of discord coincides with the geometric discord whenever local coherence vector \vec{x} vanishes. It is shown that the main difference between $D_P(\rho)$ and $D_G(\rho)$ arises from the difference between minimization of the expectation value of the matrix $\mathcal{T}\mathcal{T}^t$; The former needs to minimize expectation value of $\mathcal{T}\mathcal{T}^t$ over all $(m-1)$ -dimensional projection operators P , which can be solved exactly, but the latter needs to minimize the expectation value of the same matrix over all $(m-1)$ -dimensional simplexes $\Delta_{\{\vec{\mu}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$, which is in general a difficult task to handle.

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Appendix A: Tight lower bound on the geometric discord

In this appendix we first provide an alternative form for the geometric discord and then obtain the tight lower bound (12) in a method different from that presented in [12, 13]. Let $\{|k\rangle\}_{k=1}^m$ be any orthonormal base for \mathcal{H}^A . Following [10] we represent the projection operators corresponding to this base as

$$\Pi_k^A = |k\rangle\langle k| = \sum_{i=0}^{m^2-1} a_{ki} X_i, \quad (\text{A1})$$

where a_{ki} are defined in Eq. (6) and $k = 1, \dots, m$. It is easy to see that we can write matrix $A = (a_{ki})$ as below

$$A = \begin{pmatrix} \frac{1}{\sqrt{m}} & \vec{\mu}_1^t \\ \vdots & \vdots \\ \frac{1}{\sqrt{m}} & \vec{\mu}_m^t \end{pmatrix}, \quad (\text{A2})$$

where $\vec{\mu}_k = \frac{\sqrt{2}}{m} \vec{\alpha}_k$ with $\vec{\alpha}_k$ as defined in Eqs. (21) and (22). Therefore vectors $\{\vec{\mu}_k\}_{k=1}^m$ make the $(m-1)$ -dimensional simplex $\Delta_{\{\vec{\mu}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$. Using Eq. (11) we get

$$\text{Tr}(CC^T) = \frac{1}{mn} \left[\left(1 + \frac{2}{n} \vec{y}^t \vec{y} \right) + \frac{2}{m} \text{Tr} G \right], \quad (\text{A3})$$

and

$$\begin{aligned} [ACC^T A^T]_{kk'} &= \frac{1}{m^2 n} \left[\left(1 + \frac{2}{n} \vec{y}^t \vec{y} \right) + 2 \vec{\mu}_k^t G \vec{\mu}_{k'} \right] \\ &+ \frac{\sqrt{2}}{m^2 n} \left[\vec{\mu}_k^t \left(\vec{x} + \frac{2}{n} T \vec{y} \right) + \left(\vec{x}^t + \frac{2}{n} \vec{y}^t T^t \right) \vec{\mu}_{k'} \right] \end{aligned} \quad (\text{A4})$$

where G is defined by Eq. (13). We find therefore

$$\text{Tr}[ACC^T A^T] = \frac{1}{mn} \left[\left(1 + \frac{2}{n} \vec{y}^t \vec{y} \right) + \frac{2}{m} \sum_{k=1}^m \vec{\mu}_k^t G \vec{\mu}_k \right] \quad (\text{A5})$$

where we have used the fact that $\sum_{k=1}^m \vec{\mu}_k = \vec{0}$. Substituting Eqs. (A3) and (A5) into Eq. (3), we arrive at the following form for the geometric discord

$$D_G(\rho) = \frac{2}{m^2 n} \left[\text{Tr} G - \max_{\{\vec{\mu}_k\}} \sum_{k=1}^m \vec{\mu}_k^t G \vec{\mu}_k \right]. \quad (\text{A6})$$

Here maximum is taken over all simplexes $\Delta_{\{\vec{\mu}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$. To gain further insight into the meaning of the above equation, let us mention here that the maximization involved in the equation above can be solved exactly for the

special case $m = 2$. In this case we have to perform maximization over one dimensional simplexes $\Delta_{\{\vec{\mu}_1, \vec{\mu}_2\} \in \mathbb{R}^3}^1$ with $\vec{\mu}_1 = -\vec{\mu}_2 = \frac{1}{\sqrt{2}} \vec{\alpha}_1$ and $|\vec{\alpha}_1| = 1$. We get

$$\begin{aligned} D_G(\rho) &= \frac{1}{2n} \left[\text{Tr} G - \max_{\vec{\alpha}_1} \{ \vec{\alpha}_1^t G \vec{\alpha}_1 \} \right] \\ &= \frac{1}{2n} [\text{Tr} G - \eta_1] = \frac{1}{2n} [\eta_2 + \eta_3]. \end{aligned} \quad (\text{A7})$$

where we have defined $\eta_1 \geq \eta_2 \geq \eta_3 \geq 0$ as the eigenvalues of G . This is agree with the result obtained in Refs. [11, 14].

Unfortunately, for $m > 2$, the maximization involved in Eq. (A6) can not be solved analytically and we need to obtain lower bound. To do so, let $\{|s\rangle\}_{s=1}^m$ be the standard base of the space \mathcal{H}^A , namely the one which the $SU(m)$ generators $\{\hat{\lambda}_i^A\}_{i=1}^{m^2-1}$ are expanded in terms of them. Similar to Eq. (A1), we can write

$$\Pi_s^A = |s\rangle\langle s| = \sum_{i=0}^{m^2-1} b_{si} X_i, \quad (\text{A8})$$

where

$$b_{si} = \text{Tr}[|s\rangle\langle s| X_i] = \langle s| X_i |s\rangle, \quad (\text{A9})$$

for $s = 1, \dots, m$ and $i = 0, \dots, m^2 - 1$. Now if we choose the basis of the algebra in such a way that Cartan subalgebra makes the first $m-1$ generators, then we can write matrix $B = (b_{si})$ as follows

$$B = \begin{pmatrix} \frac{1}{\sqrt{m}} & \vec{\nu}_1^t \\ \vdots & \vdots \\ \frac{1}{\sqrt{m}} & \vec{\nu}_m^t \end{pmatrix}. \quad (\text{A10})$$

Here $\{\vec{\nu}_s\}_{s=1}^m$ are vectors in \mathbb{R}^{m^2-1} such that only first $m-1$ components of them are nonzero. So, we can write $\vec{\nu}_s = (\vec{\nu}_s, \vec{0})$ where $\{\vec{\nu}_s\}_{s=1}^m$ are vectors in \mathbb{R}^{m-1} , and $\vec{0}$ denotes null vectors in $\mathbb{R}^{m(m-1)}$. It is worth to mention that vectors $\{\vec{\nu}_s\}_{s=1}^m$ are in fact weight vectors of the $SU(m)$ Lie algebra in the defining representation [17] and satisfy the following orthonormality condition

$$\sum_{s=1}^m (\vec{\nu}_s)_k (\vec{\nu}_s)_l = \delta_{kl}. \quad (\text{A11})$$

In view of this, the zero vectors $\vec{0}$ of the definition $\vec{\nu}_s = (\vec{\nu}_s, \vec{0})$ arise from the diagonal elements of the root operators of the algebra, which are all zero. Therefore vectors $\{\vec{\nu}_s\}_{s=1}^m$ makes simplex $\Delta_{\{\vec{\nu}_s\} \in \mathbb{R}^{m^2-1}}^{m-1}$, or equivalently simplex $\Delta_{\{\vec{\nu}_s\} \in \mathbb{R}^{m-1}}^{m-1}$. Evidently, the general base $\{|k\rangle\}_{k=1}^m$ can be obtained from the standard one by a unitary transformation $U \in SU(m)$ as $\{|k\rangle\} = U\{|s\rangle\}$. Corresponding to this, there exists orthogonal transformation $\tilde{R} \in SO(m^2-1)$ such that the general simplex $\Delta_{\{\vec{\mu}_k\} \in \mathbb{R}^{m^2-1}}^{m-1}$ can be obtained from $\Delta_{\{\vec{\nu}_s\} \in \mathbb{R}^{m^2-1}}^{m-1}$, i.e.

$$(\vec{\mu}_k)_i = \sum_{j=1}^{m^2-1} \tilde{R}_{ij} (\vec{\nu}_k)_j = \sum_{j=1}^{m-1} R_{ij} (\vec{\nu}_k)_j, \quad (\text{A12})$$

for $i = 1, 2, \dots, m^2 - 1$. In the second equality $R = (R_{ij}) = (\hat{n}_j)_i$ is an $(m^2 - 1) \times (m - 1)$ left orthogonal matrix, i.e. $R^t R = I_{m-1}$, and $\hat{n}_j \in \mathbb{R}^{m^2-1}$ ($j = 1, \dots, m-1$) are orthonormal vectors, i.e. $\hat{n}_i \cdot \hat{n}_{i'} = \delta_{ii'}$. Using this and Eq. (A11), we get

$$\max_{\{\vec{\mu}_k\}} \sum_{k=1}^m \vec{\mu}_k^t G \vec{\mu}_k = \max_{\{\hat{n}_i\}} \sum_{i=1}^{m-1} \hat{n}_i^t G \hat{n}_i \leq \sum_{i=1}^{m-1} \eta_i^\downarrow, \quad (\text{A13})$$

where $\{\eta_k^\downarrow\}_{k=1}^{m^2-1}$ are eigenvalues of G in nonincreasing order. Using this in Eq. (A6), we find the desired lower bound (12) for the geometric discord. It is worth to mention that in the particular case $m = 2$, the obtained bound gives exact result for the geometric discord (see Eq. (A7)). This follows from the homomorphism $SU(2) \sim SO(3)$, happens only for $m = 2$. On the other hand, for $m > 2$ the set of all unitary transformations $U \in SU(m)$ acting on the m -dimensional Hilbert space \mathcal{H}^A will be a subset of the matrices in $SO(m^2 - 1)$. This implies that there exist rotations $\tilde{R} \in SO(m^2 - 1)$ which are not correspond to any $U \in SU(m)$, leading therefore to the inequality (A13).

Appendix B: A proof for Theorem 2

In this appendix we provide a proof for theorem 2. To this aim, we need the following lemma.

Lemma 5 (i) *If ρ is a zero-discord state on the space $\mathcal{H}^A \otimes \mathcal{H}^B$, then its corresponding local coherence vectors \vec{x} , \vec{y} , and the correlation matrix T can be represented by the following equations*

$$\vec{x} = \sum_{k=1}^m p_k \vec{\alpha}_k, \quad \vec{y} = \sum_{k=1}^m p_k \vec{\xi}_k, \quad (\text{B1})$$

$$T = \sum_{k=1}^m p_k (\vec{\alpha}_k)(\vec{\xi}_k)^t, \quad (\text{B2})$$

where $\{\vec{\alpha}_k\}_{k=1}^m$ denote coherence vectors associated to orthonormal projection operators of the subsystem A , hence satisfy Eqs. (21) and (22), but $\{\vec{\xi}_k\}_{k=1}^m$ are coherence vectors of arbitrary states of the subsystem B .

(ii) *If ρ is an arbitrary bipartite state, then its corresponding local coherence vectors \vec{x} and \vec{y} can be represented by Eq. (B1).*

Proof (i) Use the coherence vector representations for Π_k^A and ρ_k^B as

$$\Pi_k^A = \frac{1}{m} \left(\mathbb{I} + \vec{\alpha}_k \cdot \hat{\lambda}^A \right), \quad \rho_k^B = \frac{1}{n} \left(\mathbb{I} + \vec{\xi}_k \cdot \hat{\lambda}^B \right) \quad (\text{B3})$$

and insert them in the definition of zero-discord state (24). Comparing the result with the definition of ρ given in Eq. (8), one can obtain the coherence vectors \vec{x} , \vec{y} and the correlation matrix T as given by Eqs. (B1) and (B2).

(ii) Let $\rho^A = \sum_{k=1}^m p_k \Pi_k^A$, with $\{\Pi_k^A\}_{k=1}^m$ orthonormal projections on \mathcal{H}^A , be the eigenspectral decomposition of ρ^A . Then denoting coherence vectors of $\{\Pi_k^A\}_{k=1}^m$ by $\{\vec{\alpha}_k\}_{k=1}^m$, we find that $\vec{x} = \sum_{k=1}^m p_k \vec{\alpha}_k$. Now having $\{p_k\}_{k=1}^m$, we can always find set $\{\rho_k^B\}_{k=1}^m$ such that ensemble $\{p_k, \rho_k^B\}_{k=1}^m$ realizes ρ^B , i.e. $\rho^B = \sum_{k=1}^m p_k \rho_k^B$. Now letting $\{\vec{\xi}_k\}_{k=1}^m$ be coherence vectors of $\{\rho_k^B\}_{k=1}^m$, we get $\vec{y} = \sum_{k=1}^m p_k \vec{\xi}_k$. Note that for a given probability set $\{p_k\}_{k=1}^m$, states $\{\rho_k^B\}_{k=1}^m$ which realize ρ^B are not unique, so associated coherence vectors $\{\vec{\xi}_k\}_{k=1}^m$ are not unique too.

Now we are in a position to present the proof for theorem 2. If ρ is a zero-discord state, then by lemma 5 its corresponding local coherence vectors \vec{x} , \vec{y} and correlation matrix T can be represented by Eqs. (B1) and (B2), with $\{\vec{\alpha}_k\}_{k=1}^m$ as coherence vectors corresponding to orthonormal projections. Defining P as (23) and using the properties $\{\vec{\alpha}_k\}_{k=1}^m$ given in Eq. (21), one can easily shows that conditions (25) are satisfied. Conversely, we have to proof that if Eq. (25) is satisfied, then ρ is a zero-discord state, i.e. its corresponding \vec{x} , \vec{y} and T have the form given by Eqs. (B1) and (B2). To do this, we first note that Eq. (B1) is satisfied for a general state ρ . But by assumption Eq. (25) is also satisfied, leading therefore to the following form for the correlation matrix T

$$T = \sum_{k=1}^m \sum_{l=1}^m p_{kl} (\vec{\alpha}_k)(\vec{\eta}_l)^t. \quad (\text{B4})$$

Since $\{\vec{\xi}_i\}_{i=1}^m$ are not unique, we can therefore choose them in such a way that they can be expanded in terms of $\{\vec{\eta}_l\}_{l=1}^m$ as $p_k \vec{\xi}_k = \sum_{l=1}^m p_{kl} \vec{\eta}_l$. Substituting this into Eq. (B4) we get Eq. (B2), therefore \vec{x} , \vec{y} and T take the form given by Eqs. (B1) and (B2), hence ρ is a zero-discord state.

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- [1] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
 - [2] L. Henderson, and V. Vedral, J. Phys. A **34**, 6899 (2001).
 - [3] T. S. Cubitt, F. Verstraete, W. Dur, and J. I. Cirac, Phys. Rev. Lett **91**, 037902, (2003).
 - [4] B. Dakic, Y. Ole Lipp, X. Ma, M. Ringbauer, S. Kropatschek, S. Barz, T. Paterek, V. Vedral, A.

Zeilinger, C. Brukner, P. Walther, arxiv:quant-ph, 1203.1629 (2012).

- [5] K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).
- [6] A. Brodutch and D. R. Terno, Phys. Rev. A **81**, 062103 (2010).

- [7] S. Luo, Phys. Rev. A **77**, 042303 (2008); R. Dillenschneider, Phys. Rev. B **78**, 224413 (2008); M. S. Sarandy, Phys. Rev. A **80**, 022108 (2009); M. Ali, A. R. P. Rau, and G. Alber, Phys. Rev. A **81**, 042105 (2010); G. Adesso and A. Datta, Phys. Rev. Lett. **105**, 030501 (2010); P. Giorda, and M. G. A. Paris, Phys. Rev. Lett. **105**, 020503 (2010).
- [8] L. X. Cen, X. Q. Li, J. Shao, and Y. J. Yan, Phys. Rev. A **83**, 054101 (2011).
- [9] B. Dakic, V. Vedral, and C. Brukner, Phys. Rev. Lett. **105**, 190502 (2010).
- [10] S. Luo, and S. Fu, Phys. Rev. A **82**, 034302 (2010).
- [11] S. Luo, and S. Fu, Phys. Rev. Lett. **106**, 120401 (2011).
- [12] S. Rana, P. Parashar, Phys. Rev. A **85**, 024102 (2012).
- [13] A. S. M. Hassan, B. Lari, and P. S. Joag, Phys. Rev. A **85**, 024302 (2012).
- [14] Saj Vinjanampathy, and A. R. P. Rau, J. Phys. A **45**, 095303 (2012).
- [15] Xiao-Ming Lu, Jian Ma, Zhengjun Xi and Xiaoguang Wang, Phys. Rev. A **83**, 012327 (2011).
- [16] T. Zhou, J. Cui, and G. L. Long, Phys. Rev. A **84**, 062105 (2011).
- [17] H. Georgi, Lie algebra in particle physics, Advanced Book Program, (1999).
- [18] P. Rungta, V. Buzek, C. M. Caves, M. Hillery and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).